STATISTICAL RESEARCH REPORT No 9
Institute of Mathematics November 1978
University of Oslo

IMPROVED BOUNDS FOR THE AVAILABILITY AND UNAVAILABILITY IN A FIXED TIME INTERVAL FOR SYSTEMS OF MAINTAINED, INTERDEPENDENT COMPONENTS

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ABSTRACT

In this paper we arrive at a series of bounds for the availability and unavailability in the time interval $I = [t_A, t_B] \subset [0,\infty)$ for a coherent system of maintained, interdependent components. These generalize the minimal cut lower bound for the availability in $[0,t]$ given in Esary and Proschan (1970) and also most bounds for the reliability at time t given in Bodin (1970) and Barlow and Proschan (1975). In the latter special case also some new improved bounds are given. The bounds arrived at are of great interest when trying to predict the performance process of the system.

Satyanarayana and Prabhakar (1978) give a rapid algorithm for computing exact system reliability at time t. This can also be used in cases where some simpler assumptions on the dependence between the components are made. It seems, however, impossible to extend their approach to obtain exact results for the cases treated in the present paper.

1. BASIC DEFINITIONS

In the following we list a series of basic definitions which are needed. These are mostly taken from Esary and Proschan (1970) and Barlow and Proschan (1975).

We regard devices capable of two states of performance, functioning or failed, alternating between these two states in time.

Definition 1.1. The performance process of a device is a stochastic process $\{X(t), t\in\tau\}$, where for each fixed $t \in \tau, X(t)$ is a binary random variable (r.v) with

$$X(t) = \begin{cases} 1 & \text{if the device is functioning at time } t. \\ 0 & \text{if the device is failed at time } t. \end{cases}$$

The index set τ is contained in $[0,\infty)$. We assume that the sample functions $X(t),\ t \in \tau$ of a performance process are continuous from the right on τ.

<u>Definition 1.2</u>. The <u>joint performance process</u>

$$\{\underline{X}(t), t\in\tau\} = \{X_1(t),\ldots,X_n(t), t\in\tau\}$$

for a set of devices is a vector stochastic process for which the i-th marginal process $\{X_i(t), t\in\tau\}$ is the performance process for the i-th device, $i=1,\ldots,n$.

We now consider systems whose performance at each moment of time is determined by the performance of their components at that moment. Let $\{\underline{X}(t), t\in\tau\}$ be the joint performance process of the n components comprising the system. Then the performance process of the system is given by $(t\in\tau)$

$$\phi(\underline{X}(t)) = \begin{cases} 1 & \text{if the system is functioning at } t \\ 0 & \text{if the system is failed at } t, \end{cases}$$

where ϕ is the system's <u>structure function</u>. It follows that the sample functions of $\{\phi(\underline{X}(t), t\in\tau\}$ are continuous from the right on τ.

The following notation is needed.

$$(\cdot_i, \underline{x}) = (x_1,\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n).$$

<u>Definition 1.3</u>. A system is <u>coherent</u> iff

i) $\phi(\underline{x})$ is nondecreasing in each argument

ii) Each component is <u>relevant</u>; i.e.

$$\forall i, \exists(\cdot_i, \underline{x}) \ni \phi(1_i, \underline{x}) = 1 \text{ and } \phi(0_i, \underline{x}) = 0.$$

We often denote a coherent system by (C,ϕ) where C is a set of integers designating the components.

<u>Definition 1.4</u>. A <u>path set</u> is a set of components whose functioning is sufficient for the system to function. A path set is <u>minimal</u> if it can not be reduced and still be a path

set. A <u>cut set</u> is a set of components whose failure is sufficient to cause system failure. A cut set is <u>minimal</u> if it can not be reduced and still be a cut set.

We also need the following notation.

Let $A \subseteq C$. Then

$\underline{x}^A$ = vector with elements $x_i, i \in A$

A^c = subset of C complementary to A.

<u>Definition 1.5.</u> The <u>minimal path series stucture</u>, $\rho(\underline{x}^P)$, corresponding to the minimal path set, P, is given by

$$\rho(\underline{x}^P) = \prod_{i \in P} x_i .$$

Similarly, the <u>minimal cut parallel structure</u>, $\kappa(\underline{x}^K)$, corresponding to the minimal cut set, K, is given by

$$\kappa(\underline{x}^K) = \coprod_{i \in K} x_i \overset{\text{def}}{=} 1 - \prod_{i \in K} (1 - x_i).$$

<u>Definition 1.6.</u> The coherent system (A, χ) is a <u>module</u> of the coherent system (C, ϕ) iff

$$\phi(\underline{x}) = \psi[\chi(\underline{x}^A), \underline{x}^{A^c}],$$

where ψ is a coherent structure function and $A \subseteq C$.

Intuitively, a module is a coherent subsystem that acts as if it were just a component.

<u>Definition 1.7.</u> A <u>modular decomposition</u> of a coherent system (C, ϕ) is a set of disjoint modules $\{(A_k, \chi_k)\}_{k=1}^r$ together with an organizing <u>coherent structure</u> ψ; i.e.

i) $C = \bigcup_{i=1}^{r} A_i$ where $A_i \cap A_j = \emptyset \quad i \neq j$

ii) $\phi(\underline{x}) = \psi[\chi_1(\underline{x}^{A_1}),\ldots,\chi_r(\underline{x}^{A_r})]$.

Definition 1.8. Given a coherent structure ϕ, its dual structure ϕ^D is given by

$$\phi^D(\underline{x}) = 1 - \phi(\underline{1}-\underline{x}),$$

where $\underline{1} - \underline{x} = (1-x_1,\ldots,1-x_n)$.

Consider a time interval $I = [t_A,t_B] \subset [0,\infty)$ and let $\tau(I) = \tau \cap I$.

Definition 1.9. The marginal performance processes $\{X_i(t), t\in\tau\}$ $i=1,\ldots,n$ for a set of devices are independent in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset \tau(I)$, the random vectors

$$(X_1(t_1),\ldots,X_1(t_m)),\ldots,(X_n(t_1),\ldots,X_n(t_m))$$

are independent.

Definition 1.10. A modular decomposition $\{A_i\chi_i\}_{i=1}^r$ consists of totally independent modules in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset \tau(I)$, the random vectors

$$(\underline{X}^{A_1}(t_1),\ldots,\underline{X}^{A_1}(t_m)),\ldots,(\underline{X}^{A_r}(t_1),\ldots,\underline{X}^{A_r}(t_m))$$

are independent.

The latter definition seems to be new.

Definition 1.11. The r.v.'s $T_1,\ldots,T_n$ are associated iff $\mathrm{Cov}[\Gamma(T),\Delta(\underline{T})] \geq 0$ for all pairs of nondecreasing binary functions Γ,Δ.

We list some basic properties of associated r.v.'s:

P_1) Any subset of a set of associated r.v.'s is a set of associated r.v.'s.

P_2) The set consisting of a single r.v. is a set of associated r.v.'s.

P_3) Nondecreasing functions of associated r.v.'s are associated.

P_4) If two sets of associated r.v.'s are independent of each other, then their union is a set of associated r.v.'s.

Definition 1.12. The joint performance process $\{\underline{X}(t), t \in \tau\}$ for a set of devices is associated in the time interval I iff, for any integer m and $\{t_1, \ldots, t_m\} \subset \tau(I)$, the r.v.'s in the array

$$\begin{matrix} X_1(t_1), \ldots\ldots, X_1(t_m) \\ \vdots \qquad\qquad \vdots \\ X_n(t_1), \ldots\ldots, X_n(t_m) \end{matrix}$$

are associated.

This definition obviously applies to a marginal performance process too.

Definition 1.13. The availability, $h_\phi^{(I)}$, and the unavailability, $g_\phi^{(I)}$, in the time interval I for a coherent system ϕ are given by

$$h_\phi^{(I)} = P[\phi(\underline{X}(s)) = 1 \quad \forall\, s \in \tau(I)]$$
$$g_\phi^{(I)} = P[\phi(\underline{X}(s)) = 0 \quad \forall\, s \in \tau(I)]$$

Assume that all components and hence the system function at $t = 0$, but are not maintained.

Definition 1.14. The reliability at time t, h_ϕ, for a coherent system ϕ is given by

$$h_\phi = h_\phi^{[t,t]}$$

Note that we obviously have

$$h_\phi = h_\phi^{[0,t]} = 1 - g_\phi^{[t,t]}. \tag{1.1}$$

In the present paper we will arrive at upper and lower bounds for $h_\phi^{(I)}$ and $g_\phi^{(I)}$ in the case of maintained, interdependent components. These bounds are of great interest when trying to predict the performance process of the system. Our results generalize the minimal cut lower bound given in Esary and Proschan (1970) for the case $I = [0,t]$, and also most bounds given in Bodin (1970) and Barlow and Proschan (1975) for $I = [t,t]$ and devices which are not maintained. For the general case our results are given in Section 2. In the latter special case some new improved bounds are given in Section 3.

It should finally be noted that Satyanarayana and Prabhakar (1978) give a rapid algorithm for computing exact system reliability at time t. This can also be used in cases where some simpler assumptions on the dependence between the components are made. It seems, however, impossible to extend their approach to obtain exact results for the cases treated in the present paper.

2. IMPROVED BOUNDS FOR THE AVAILABILITY AND UNAVAILABILITY IN A FIXED TIME INTERVAL FOR SYSTEMS OF MAINTAINED, INTERDEPENDENT COMPONENTS

The following theorem is a slight generalization of the first part of Theorem 3.9 in Chapter 2 of Barlow and Proschan (1975). (All references to this book in the following concern this chapter if nothing to the contrary is said.) The proof is almost identical and is left to the reader.

Theorem 2.1. Let (C,ϕ) be a coherent system with minimal path sets $P_1,\ldots,P_p$ and minimal cut sets $K_1,\ldots,K_k$. Define

$$l''_\phi{}^{(I)} = \max_{1\le j\le p} P[\min_{i\in P_j} X_i(s) = 1 \quad \forall\, s\in \tau(I)]$$

$$u''_\phi{}^{(I)} = \min_{1\le j\le k} P[\max_{i\in K_j} X_i(s) = 1 \quad \forall\, s\in \tau(I)]$$

$$\bar{l}''_\phi{}^{(I)} = \max_{1\le j\le k} P[\max_{i\in K_j} X_i(s) = 0 \quad \forall\, s\in \tau(I)]$$

$$\bar{u}''_\phi{}^{(I)} = \min_{1\le j\le p} P[\min_{i\in P_j} X_i(s) = 0 \quad \forall\, s\in \tau(I)].$$

Then

$$l''_\phi{}^{(I)} \le h_\phi{}^{(I)} \le u''_\phi{}^{(I)}$$

$$\bar{l}''_\phi{}^{(I)} \le g_\phi{}^{(I)} \le \bar{u}''_\phi{}^{(I)}$$

Theorem 2.1 is very general, but seems of little practical value due to the complexity of the bounds. The following corollary is a generalization of the second part of the mentioned theorem in Barlow and Proschan (1975). The proof is inspired by the one for the minimal cut lower bound in Esary and Proschan (1970). This is true for a series of proofs given in this paper.

In the following we denote the availability and unavailability in I for the i-th component of a coherent system (C,ϕ) by $p_i^{(I)}$ and $q_i^{(I)}$ respectively, $i=1,\ldots,n$ and let

$$\underline{p}^{(I)} = (p_1^{(I)},\ldots,p_n^{(I)}), \quad \underline{q}^{(I)} = (q_1^{(I)},\ldots,q_n^{(I)}).$$

<u>Corollary 2.2</u>. Consider the situation in the preceding theorem, but assume furthermore that the joint performance process of the system's components is associated in the time interval I. Define

$$l'_\phi(\underline{p}^{(I)}) = \max_{1\le j\le p} \prod_{i\in P_j} p_i^{(I)}$$

$$\bar{l}'_\phi(\underline{q}^{(I)}) = \max_{1\le j\le k} \prod_{i\in K_j} q_i^{(I)} .$$

Then

$$l'_\phi(\underline{p}^{(I)}) \le h_\phi^{(I)} \le 1-\bar{l}'_\phi(\underline{q}^{(I)}) \tag{2.1}$$

$$\bar{l}'_\phi(\underline{q}^{(I)}) \le g_\phi^{(I)} \le 1-l'_\phi(\underline{p}^{(I)}). \tag{2.2}$$

<u>Proof</u>: Let S be a countable subset of $\tau(I)$ that is dense in $\tau(I)$. Since the sample functions of $\{X_i(t),t\in\tau\}$ $i=1,\ldots,n$ are continuous from the right, then

$$l''^{(I)}_\phi = \max_{1\le j\le p} P[X_i(s) = 1 \quad \forall\, i\in P_j \text{ and } \forall\, s\in S].$$

Let $S_m = \{s_1,\ldots,s_m\}$ $m=1,2,\ldots$ be subsets of S such that $S_m \nearrow S$ as $m\to\infty$. By monotone convergence

$$\max_{1\le j\le p} P[X_i(s) = 1 \quad \forall\, i\in P_j \text{ and } \forall\, s\in S_m] \searrow l''^{(I)}_\phi .$$

Now

$$\max_{1\le j\le p} P[X_i(s) = 1 \quad \forall\, i \in P_j \text{ and } \forall\, s \in S_m]$$

$$= \max_{1\le j\le p} P[\prod_{i\in P_j} \prod_{l=1}^{m} X_i(s_l) = 1]$$

$$\ge \max_{1\le j\le p} \prod_{i\in P_j} P[X_i(s) = 1 \quad \forall\, s \in S_m],$$

having applied Theorem 3.1 in Barlow and Proschan (1975). This can be done since the r.v.'s $\prod_{l=1}^{m} X_i(s_l)$ $i \in P_j$ are associated by property P_3. Finally by monotone convergence

$$\max_{1\le j\le p} \prod_{i\in P_j} P[X_i(s) = 1 \quad \forall\, s \in S_m] \searrow l_\phi^!(\underline{p}^{(I)}).$$

From Theorem 2.1 the left inequality of (2.1) follows.

Since $K_1,\ldots,K_k$ are the minimal path sets of the dual structure ϕ^D, the latter inequality applied on ϕ^D gives

$$\max_{1\le j\le k} \prod_{i\in K_j} q_i^{(I)} \le h_{\phi^D}^{(I)} = g_\phi^{(I)},$$

and the left inequality of (2.2) is proved. The two upper bounds in the corollary follow from the two lower bounds by noting that

$$h_\phi^{(I)} + g_\phi^{(I)} \le 1,$$

and the proof is completed.

Note that these upper bounds are poor if $h_\phi^{(I)} + g_\phi^{(I)}$ is not close to 1. We have, however, not been able to arrive at better upper bounds.

The next theorem is a generalization of Theorem 3.4 in Barlow and Proschan (1975), the lower bound for $h_\phi^{(I)}$ being the minimal cut lower bound, given in Esary and Proschan (1970), applied on a general interval I instead on $[0,t]$.

Theorem 2.3. Let (C,ϕ) be a coherent system with joint performance process for its components being associated in the time interval I. Let $\rho_1(\underline{x}^{P_1}),\ldots,\rho_p(\underline{x}^{P_p})$ be the minimal path series structures and $\kappa_1(\underline{x}^{K_1}),\ldots,\kappa_k(\underline{x}^{K_k})$ be the minimal cut parallel structures of (C,ϕ). Define

$$l_\phi^{*(I)} = \prod_{j=1}^{k} h_{\kappa_j}^{(I)}$$

$$\bar{l}_\phi^{*\,(I)} = \prod_{j=1}^{p} g_{\rho_j}^{(I)}$$

Then

$$l_\phi^{*(I)} \leq h_\phi^{(I)} \leq 1 - \bar{l}_\phi^{*(I)} \tag{2.3}$$

$$\bar{l}_\phi^{*(I)} \leq g_\phi^{(I)} \leq 1 - l_\phi^{*(I)}. \tag{2.4}$$

Proof: The proof of the left inequality of (2.3) is a trivial extension of the proof of the mentioned lower bound in Esary and Proschan (1970) and is left to the reader. Since $\rho_1^D(\underline{x}^{P_1}),\ldots,\rho_p^D(\underline{x}^{P_p})$ are the minimal cut parallel structures of the dual structure ϕ^D, the latter inequality applied on ϕ^D gives

$$\prod_{j=1}^{p} g_{\rho_j}^{(I)} = \prod_{j=1}^{p} h_{\rho_j^D}^{(I)} \leq h_{\phi^D}^{(I)} = g_\phi^{(I)},$$

and the left inequality of (2.4) is proved. The two upper bounds in the theorem again follow from the two lower bounds and the proof is completed.

Note again that these upper bounds can be very poor. By combining Corollary 2.2 and Theorem 2.3 we arrive at the following corollary

Corollary 2.4. Make the same assumptions as in Theorem 2.3 and define

$$L_\phi^{(I)} = \max(l'_\phi(\underline{p}^{(I)}), l^{*(I)}_\phi)$$
$$\bar{L}_\phi^{(I)} = \max(\bar{l}'_\phi(\underline{q}^{(I)}), \bar{l}^{*(I)}_\phi).$$

Then

$$L_\phi^{(I)} \leq h_\phi^{(I)} \leq 1 - \bar{L}_\phi^{(I)}$$
$$\bar{L}_\phi^{(I)} \leq g_\phi^{(I)} \leq 1 - L_\phi^{(I)}.$$

Still we can not guarantee the upper bounds to be very useful. Another objection against the bounds given is that $l^{*(I)}_\phi$ and $\bar{l}^{*(I)}_\phi$ seem very complex. The latter objection is dealt with in the next corollary, the price paid being stronger assumptions. The corollary generalizes a result in Barlow and Proschan (1975).

Corollary 2.5. Let (C,ϕ) be a coherent system with the marginal performance processes of its components being mutually independent and each of them associated in the time interval I. Define

$$l^*_\phi(\underline{p}^{(I)}) = \prod_{j=1}^{k} \coprod_{i \in K_j} p_i^{(I)}$$
$$\bar{l}^*_\phi(\underline{q}^{(I)}) = \prod_{j=1}^{p} \coprod_{i \in P_j} q_i^{(I)},$$

where $P_1,\ldots,P_p$ are the minimal path sets and $K_1,\ldots,K_k$ the minimal cut sets of (C,ϕ). Furthermore, introduce

$$L_\phi(\underline{p}^{(I)}) = \max(l'_\phi(\underline{p}^{(I)}), l^*_\phi(\underline{p}^{(I)}))$$
$$\bar{L}_\phi(\underline{q}^{(I)}) = \max(\bar{l}'_\phi(\underline{q}^{(I)}), \bar{l}^*_\phi(\underline{q}^{(I)})).$$

Then

$$L_\phi(\underline{p}^{(I)}) \leq L_\phi^{(I)} \leq h_\phi^{(I)} \leq 1-\bar{L}_\phi^{(I)} \leq 1-\bar{L}_\phi(\underline{q}^{(I)}) \qquad (2.5)$$

$$\bar{L}_\phi(\underline{q}^{(I)}) \leq \bar{L}_\phi^{(I)} \leq g_\phi^{(I)} \leq 1-L_\phi^{(I)} \leq 1-L_\phi(\underline{p}^{(I)}) \qquad (2.6)$$

Proof: By property P_4 for associated r.v.'s the joint performance process for the system's components is associated in I. Hence Corollary 2.4 is applicable. What remains is then to prove the two inequalities most to the left. Introduce the sets S_m, m=1,2,... as before. Now

$$\begin{aligned}
&P[\kappa_j(\underline{X}(s)) = 1 \quad \forall s \in S_m] \\
&= P[\forall s \in S_m \, \exists\, i \in K_j \ni X_i(s) = 1] \\
&\geq P[\exists i \in K_j \ni X_i(s) = 1 \quad \forall s \in S_m] \\
&= 1 - P[\forall i \in K_j \, \exists\, s \in S_m \ni X_i(s) = 0] \\
&= 1 - P[\prod_{i \in K_j} (1 - \prod_{l=1}^{m} X_i(s_l)) = 1] \\
&= 1 - \prod_{i \in K_j} P[\exists s \in S_m \ni X_i(s) = 0] \\
&= \coprod_{i \in K_j} P[X_i(s) = 1 \quad \forall s \in S_m],
\end{aligned}$$

having applied the independence of the marginal performance processes of the components. By letting $m \to \infty$, we get

$$l_\phi^{*(I)} \geq l_\phi^*(\underline{p}^{(I)}).$$

Applying the latter inequality on ϕ^D, we get

$$\bar{l}_\phi^{*(I)} = \prod_{j=1}^{p} h_{\rho_j^D}^{(I)} \geq \prod_{j=1}^{p} \coprod_{i \in P_j} q_i^{(I)} = \bar{l}_\phi^*(\underline{q}^{(I)}) .$$

Having established these inequalities the proof is completed.

Inspired by Bodin (1970) we finally give two theorems where new bounds are obtained using modular decompositions. Some of these bounds are proved to be improvements of bounds already given.

<u>Theorem 2.6</u>. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^{r}$ consisting of totally independent modules in the time interval I and with organizing coherent structure function ψ. Furthermore, assume that the marginal performance process of each module is associated in I. Then

$$1-\bar{L}_{\psi}[\bar{l}''^{(I)}_{\chi_1},\dots,\bar{l}''^{(I)}_{\chi_r}] \overset{1}{\geq} 1-\bar{L}_{\psi}[g^{(I)}_{\chi_1},\dots,g^{(I)}_{\chi_r}]$$

$$\overset{2}{\geq} h^{(I)}_{\phi} \overset{3}{\geq} L_{\psi}[h^{(I)}_{\chi_1},\dots,h^{(I)}_{\chi_r}]$$

$$\overset{4}{\geq} L_{\psi}[l''^{(I)}_{\chi_1},\dots,l''^{(I)}_{\chi_r}] \overset{5}{\geq} l''^{(I)}_{\phi}$$

$$1-\bar{l}'_{\psi}[\bar{l}''^{(I)}_{\chi_1},\dots,\bar{l}''^{(I)}_{\chi_r}] \overset{6}{\geq} u''^{(I)}_{\phi}$$

$$1-L_{\psi}[l''^{(I)}_{\chi_1},\dots,l''^{(I)}_{\chi_r}] \overset{7}{\geq} 1-L_{\psi}[h^{(I)}_{\chi_1},\dots,h^{(I)}_{\chi_r}]$$

$$\overset{8}{\geq} g^{(I)}_{\phi} \overset{9}{\geq} \bar{L}_{\psi}[g^{(I)}_{\chi_1},\dots,g^{(I)}_{\chi_r}]$$

$$\overset{10}{\geq} \bar{L}_{\psi}[\bar{l}''^{(I)}_{\chi_1},\dots,\bar{l}''^{(I)}_{\chi_r}] \overset{11}{\geq} \bar{l}''^{(I)}_{\phi}$$

$$1-l'_{\psi}[l''^{(I)}_{\chi_1},\dots,l''^{(I)}_{\chi_r}] \overset{12}{\geq} \bar{u}''^{(I)}_{\phi} .$$

Note that we have found lower bounds that are improvements of the lower bounds in Theorem 2.1. This is by no means proved with regard to the upper bounds.

Obviously, the inequalities 1 and 10 are equivalent. The same is true for 4 and 7. From the assumptions of the theorem it follows that the marginal performance processes of the modules of (C,ϕ) are mutually independent in I. Hence Corollary 2.5 is applicable and the inequalities 2,3,8,9 follow. Furthermore, Theorem 2.1 can be applied on all modules. Hence the inequalities 1 and 4 follow since $\bar{L}_\psi$ and L_ψ are non-decreasing functions in each argument. The remaining inequalities are proved in Appendix I.

Theorem 2.7. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^{r}$ and with organizing coherent structure function ψ. Assume the marginal performance processes of the modules to be mutually independent in I and furthermore the joint performance process for the components of each module to be associated in I. Then

$$1-\bar{l}'_\phi(\underline{q}^{(I)}) \overset{1}{\geq} 1-\bar{L}_\psi[\bar{L}_{\chi_1}^{(I)},\ldots,\bar{L}_{\chi_r}^{(I)}]$$

$$\overset{2}{\geq} 1-\bar{L}_\psi[g_{\chi_1}^{(I)},\ldots,g_{\chi_r}^{(I)}] \overset{3}{\geq} h_\phi^{(I)}$$

$$\overset{4}{\geq} L_\psi[h_{\chi_1}^{(I)},\ldots,h_{\chi_r}^{(I)}] \overset{5}{\geq} L_\psi[L_{\chi_1}^{(I)},\ldots,L_{\chi_r}^{(I)}]$$

$$\overset{6}{\geq} l'_\phi(\underline{p}^{(I)})$$

$$1-l'_\phi(\underline{p}^{(I)}) \overset{7}{\geq} 1-L_\psi[L_{\chi_1}^{(I)},\ldots,L_{\chi_r}^{(I)}]$$

$$\overset{8}{\geq} 1-L_\psi[h_{\chi_1}^{(I)},\ldots,h_{\chi_r}^{(I)}] \overset{9}{\geq} g_\phi^{(I)}$$

$$\overset{10}{\geq} \bar{L}_\psi[g_{\chi_1}^{(I)},\ldots,g_{\chi_r}^{(I)}] \overset{11}{\geq} \bar{L}_\psi[\bar{L}_{\chi_1}^{(I)},\ldots,\bar{L}_{\chi_r}^{(I)}]$$

$$\overset{12}{\geq} \bar{l}'_\phi(\underline{q}^{(I)}) .$$

Note that we have found bounds that are improvements of the bounds in Corollary 2.2. (We have also proved this corollary under somewhat different assumptions.) However, we have not proved that these bounds are improvements of the ones given in Corollary 2.4.

Obviously, the inequalities 1 and 12, 2 and 11, 5 and 8 and finally 6 and 7 are equivalent. From the assumptions of the theorem and property P_3 of associated r.v.'s it follows that the marginal performance process of each module of (C,ϕ) is associated in I. Hence Corollary 2.5 is applicable and the inequalities 3,4,9,10 follow. Furthermore, Corollary 2.4 can be applied on all modules. Hence the inequalities 2 and 5 follow since $\bar{L}_\psi$ and L_ψ are nondecreasing functions in each argument. The remaining inequalities are proved in Appendix II.

3. IMPROVED BOUNDS FOR THE RELIABILITY AT A FIXED POINT OF TIME FOR SYSTEMS OF INTERDEPENDENT COMPONENTS THAT ARE NOT MAINTAINED

In this section we assume that all components and hence the system function at $t=0$, but are not maintained. Furthermore $I = [t,t]$. In the following we obtain bounds for the reliability at time t, h_ϕ, by using modular decompositions. For simplicity all bounds introduced in the previous section will for the special case treated here appear without "I".

<u>Theorem 3.1</u>. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^r$ consisting of totally independent modules at time t and with organizing coherent structure function ψ. Then

$$
\begin{array}{lll}
1-\bar{I}''_{\phi} \overset{1}{\geq} 1-\bar{L}_{\psi}[\bar{I}''_{\chi_1},\ldots,\bar{I}''_{\chi_r}] & & \\
\overset{2}{\geq} 1-\bar{L}_{\psi}[1-h_{\chi_1},\ldots,1-h_{\chi_r}] \overset{4}{\geq} & & \\
 & h_{\phi} & \\
\overset{3}{\geq} h_{\psi}[1-\bar{I}''_{\chi_1},\ldots,1-\bar{I}''_{\chi_r}] \quad \overset{5}{\geq} & & \\
\overset{6}{\geq} L_{\psi}[h_{\chi_1},\ldots,h_{\chi_r}] \overset{8}{\geq} & & \\
 & L_{\psi}[l''_{\chi_1},\ldots,l''_{\chi_r}] & \\
\overset{7}{\geq} h_{\psi}[l''_{\chi_1},\ldots,l''_{\chi_r}] \overset{9}{\geq} & & \\
\overset{10}{\geq} l''_{\phi} \,. & &
\end{array}
$$

<u>Proof</u>: The inequalities 1,2,4,6,8,10 follow immediately from the inequalities 11,10,9,3,4,5 of Theorem 2.6 by specializing for the case treated in this section. To apply the latter theorem we need the marginal performance process of each module to be associated at time t. This follows from property P_2 for associated r.v.'s. Note that we can write

$$h_{\phi} = h_{\psi}[h_{\chi_1},\ldots,h_{\chi_r}], \tag{3.1}$$

where h_{ψ} is a nondecreasing function in each argument. Hence the inequalities 5 and 7 follow from Theorem 2.1 and the inequalities 3 and 9 from Corollary 2.5.

Note that for the special case treated in this section (1.1) is valid, and hence we have no objections against the upper bounds arrived at in the previous section, when applied here. However, any upper bound arrived at is now equivalent to an established lower bound and nothing is gained. Note also that Theorem 2.6 can be considered as a generalization of the main part of Theorem 3.1. For the remaining theorems given in this section, we have not been able to generalize their main parts to the situation treated in Section 2.

<u>Theorem 3.2</u>. Make the same assumptions as in Theorem 3.1. Assume furthermore that for each module the states of the components at time t are associated r.v.'s. Then

$$
\begin{array}{lll}
1-\bar{L}_\phi \overset{1}{\geq} 1-\bar{L}_\psi[\bar{L}_{\chi_1},\ldots,\bar{L}_{\chi_r}] & & \\
\overset{2}{\geq} 1-\bar{L}_\psi[1-h_{\chi_1},\ldots,1-h_{\chi_r}] & \overset{4}{\geq} & \\
 & & h_\phi \\
\overset{3}{\geq} h_\psi[1-\bar{L}_{\chi_1},\ldots,1-\bar{L}_{\chi_r}] & \overset{5}{\geq} & \\
\overset{6}{\geq} L_\psi[h_{\chi_1},\ldots,h_{\chi_r}] & \overset{8}{\geq} & \\
 & & L_\psi[L_{\chi_1},\ldots,L_{\chi_r}] \\
\overset{7}{\geq} h_\psi[L_{\chi_1},\ldots,L_{\chi_r}] & \overset{9}{\geq} & \\
\overset{10}{\geq} L_\phi . & &
\end{array}
$$

The inequalities 2,4,6,8 follow immediately from the inequalities 2,3,4,5 of Theorem 2.7 by specializing for the case treated in this section. Rebembering (3.1) the inequalities 5 and 7 follow from Corollary 2.4 and the inequalities 3 and 9 from Corollary 2.5. The remaining two inequalities are proved in Appendix III.

Theorem 3.2 gives improved bounds compared with the ones given in Theorem 3.4 and 3.9 in Barlow and Proschan (1975), which were generalized in Corollary 2.4. (The bounds of these theorems are in fact proved under somewhat different assumptions here.)

Note that even for the special case where the states of the components are mutually independent at time t, all the inequalities of Theorem 3.2 are generalizations of the Theorems 2 and 6 given in Bodin (1970). The latter ones are obtained by replacing, for any coherent system (B,Ω), $\bar{L}_\Omega(\underline{q}^B)$ and $\bar{L}_\Omega$ by $\bar{l}^*_\Omega(\underline{q}^B)$, and $L_\Omega(\underline{p}^B)$ and L_Ω by $l^*_\Omega(\underline{p}^B)$ (see Corollary 2.5). One should, however, remember that for this special case exact results can be obtained by the algorithm of Satyanarayana and Prabhakar (1978).

We will finally give two theorems which under certain assumptions essentially tell us that it is advantageous to decompose modules with unknown reliabilities at time t and do nothing with the remaining ones. The theorems are inspired by the Theorems 3 and 7 in Bodin (1970).

Theorem 3.3. Make the same assumptions as in Theorem 3.1. Furthermore, for $i=1,\ldots,k$ $(1 \leq k \leq r)$ assume that (A_i,χ_i) has a modular decomposition $\{(B_{ij},\Omega_{ij})\}_{j=1}^{s_i}$ consisting of totally independent modules at time t and with organizing coherent structure function σ_i. Introduce

$$\Theta(\underline{\Omega}) = \psi[\sigma_1(\Omega_{11},\ldots,\Omega_{1s_1}),\ldots,\sigma_k(\Omega_{k1},\ldots,\Omega_{ks_k}),\chi_{k+1},\ldots,\chi_r],$$

where

$$\underline{\Omega} = (\Omega_{11},\ldots,\Omega_{1s_1},\Omega_{21},\ldots,\Omega_{ks_k},\chi_{k+1},\ldots,\chi_r) .$$

Then

$$1 - \bar{L}_\psi[1-h_{\chi_1},\ldots,1-h_{\chi_k},\bar{l}''_{\chi_{k+1}},\ldots,\bar{l}''_{\chi_r}]$$

1

$$\leq 1 - \bar{L}_\Theta[1-h_{\Omega_{11}},\ldots,1-h_{\Omega_{ks_k}},\bar{l}''_{\chi_{k+1}},\ldots,\bar{l}''_{\chi_r}]$$

$$h_\psi[1-\bar{l}''_{\chi_1},\ldots,1-\bar{l}''_{\chi_k},h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$

$$\stackrel{2}{\geq} h_\Theta[1-\bar{I}''_{\Omega_{11}},\ldots,1-\bar{I}''_{\Omega_{ks_k}},h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$

$$L_\psi[h_{\chi_1},\ldots,h_{\chi_k},l''_{\chi_{k+1}},\ldots,l''_{\chi_r}]$$
$$\stackrel{3}{\geq} L_\Theta[h_{\Omega_{11}},\ldots,h_{\Omega_{ks_k}},l''_{\chi_{k+1}},\ldots,l''_{\chi_r}]$$

$$h_\psi[l''_{\chi_1},\ldots,l''_{\chi_k},h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$
$$\stackrel{4}{\leq} h_\Theta[l''_{\Omega_{11}},\ldots,l''_{\Omega_{ks_k}},h_{\chi_{k+1}},\ldots,h_{\chi_r}].$$

<u>Proof</u>: Introduce $(i=1,\ldots,k\,;\ j=1,\ldots,s_i)$

$$p_{ij} = P[\Omega_{ij}(\underline{X}^{B_{ij}})=1] = h_{\Omega_{ij}}$$

$$\underline{p}_i = (p_{i1},\ldots,p_{is_i})$$

$$\underline{p} = (\underline{p}_1,\ldots,\underline{p}_k,1-\bar{I}''_{\chi_1},\ldots,1-\bar{I}''_{\chi_r})\ .$$

Then

$$1-\bar{L}_\psi[1-h_{\chi_1},\ldots,1-h_{\chi_k},\bar{I}''_{\chi_1},\ldots,\bar{I}''_{\chi_r}]$$

$$\leq 1-\bar{L}_\Theta \leq 1-\bar{L}_\Theta(\underline{1}-\underline{p})\ ,$$

having applied Theorem 3.2 and Corollary 2.5. Hence the first inequality is established. From Theorem 3.1 we have $(i=1,\ldots,k)$

$$1-\bar{I}''_{\chi_i} \geq h_{\sigma_i}[1-\bar{I}''_{\Omega_{i1}},\ldots,1-\bar{I}''_{\Omega_{is_i}}].$$

Since h_ψ is nondecreasing in each argument, we have

$$h_\psi[1-\bar{I}''_{\chi_1},\ldots,1-\bar{I}''_{\chi_k},h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$

$$\geq h_\psi[h_{\sigma_1}(1-\bar{I}''_{\Omega_{11}},\ldots,1-\bar{I}''_{\Omega_{1s_1}}),\ldots,$$
$$h_{\sigma_k}(1-\bar{I}''_{\Omega_{k1}},\ldots,1-\bar{I}''_{\Omega_{ks_k}}),h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$

$$= h_\Theta[1-\bar{I}''_{\Omega_{11}},\ldots,1-\bar{I}''_{\Omega_{ks_k}},h_{\chi_{k+1}},\ldots,h_{\chi_r}],$$

and the second inequality is proved. The remaining inequalities are proved similarly.

Note that from the Theorems 2.1 and 3.1 we have

$$1 - \bar{L}_\psi[1-h_{\chi_1},\dots,1-h_{\chi_k},\bar{I}''_{\chi_{k+1}},\dots,\bar{I}''_{\chi_r}]$$
$$\geq 1-\bar{L}_\psi[1-h_{\chi_1},\dots,1-h_{\chi_r}] \geq h_\phi .$$

Similarly from Theorem 2.1 we have

$$h_\Theta[1-\bar{I}''_{\Omega_{11}},\dots,1-\bar{I}''_{\Omega_{ks_k}},h_{\chi_{k+1}},\dots,h_{\chi_r}]$$
$$\geq h_\Theta[h_{\Omega_{11}},\dots,h_{\Omega_{ks_k}},h_{\chi_{k+1}},\dots,h_{\chi_r}]$$
$$= h_\phi \ .$$

This shows that the two first relations of the theorem give upper bounds for h_ϕ . Analogously it follows that the two last relations of the theorem give lower bounds for h_ϕ.

<u>Theorem 3.4</u>. Make the same assumptions as in Theorem 3.3. Furthermore, assume that for each of the modules (B_{ij},Ω_{ij}) $i=1,\dots,k$; $j=1,\dots,s_i$ and (A_i,χ_i) $i=k+1,\dots,r$ the states of the components at time t are associated r.v.'s. Then

$$1 - \bar{L}_\psi[1-h_{\chi_1},\dots,1-h_{\chi_k},\bar{L}_{\chi_{k+1}},\dots,\bar{L}_{\chi_r}]$$
$$\overset{1}{\leq} 1 - \bar{L}_\Theta[1-h_{\Omega_{11}},\dots,1-h_{\Omega_{ks_k}},\bar{L}_{\chi_{k+1}},\dots,\bar{L}_{\chi_r}]$$

$$h_\psi[1-\bar{L}_{\chi_1},\dots,1-\bar{L}_{\chi_k},h_{\chi_{k+1}},\dots,h_{\chi_r}]$$
$$\overset{2}{\geq} h_\Theta[1-\bar{L}_{\Omega_{11}},\dots,1-\bar{L}_{\Omega_{ks_k}},h_{\chi_{k+1}},\dots,h_{\chi_r}]$$

$$L_\psi[h_{\chi_1},\ldots,h_{\chi_k},L_{\chi_{k+1}},\ldots,L_{\chi_r}]$$
$$\overset{3}{\geq} L_\Theta[h_{\Omega_{11}},\ldots,h_{\Omega_{ks_k}},L_{\chi_{k+1}},\ldots,L_{\chi_r}]$$
$$h_\psi[L_{\chi_1},\ldots,L_{\chi_k},h_{\chi_{k+1}},\ldots,h_{\chi_r}]$$
$$\overset{4}{\leq} h_\Theta[L_{\Omega_{11}},\ldots,L_{\Omega_{ks_k}},h_{\chi_{k+1}},\ldots,h_{\chi_r}],$$

where the coherent structure Θ is defined in Theorem 3.3.

Proof: The proof is very similar to the one of Theorem 3.3 applying Theorem 3.2 instead of Theorem 3.1. In addition the bounds which were generalized in Corollary 2.4 are used.

Again it can be shown that the two first relations of the theorem give upper bounds for h_ϕ whereas the two last ones give lower bounds.

Note that the first and third inequality of the two latter theorems trivially can be replaced by equalities if $1-\bar{L}_\psi$ and L_ψ are replaced by h_ψ, and $1-\bar{L}_\Theta$ and L_Θ by h_Θ (just remember (3.1)). These four inequalities and the additional equalities give us the following information.

Consider a coherent system (C,ϕ) with modular decomposition consisting of totally independent modules at time t and with organizing coherent structure function ψ having unknown (known) reliability function h_ψ. (This is considered as a function of the reliabilities of the modules - again remember (3.1).) Furthermore, assume that each module with known reliability has a modular decomposition consisting of modules having either the specific properties mentioned in Theorem 3.3 or 3.4 and in addition having known reliabilities at time t. Finally, assume that the organizing coherent structure function Θ of the refined modular decomposition of (C,ϕ) has unknown (known) reliability function h_Θ. Then the bounds based on the latter modular decomposition are worse than (just as good as) the bounds based on the former.

The remaining inequalities of the two latter theorems tell us in a way just the opposite. Now both h_ψ and h_Θ are supposed to be known. However, this time each module with unknown reliability has a modular decomposition consisting of modules having either the specific properties mentioned in Theorem 3.3 or 3.4 and in addition having unknown reliabilities at time t. Now the bounds based on the refined modular decomposition are the better.

4. SOME COMMENTS ON FUTURE RESEARCH

For the general case treated in Section 2 we have not been able to arrive at satisfactory upper bounds. This is obviously an area for future research. Secondly, we would like to generalize the main parts of the Theorems 3.2, 3.3 and 3.4.

In a way the main part of Theorem 3.2 is the inequalities 1,2,4,6,8,10 out of which all except 1 and 10 are specializations of inequalities of Theorem 2.7. Hence what we would like to prove is the following

Conjecture 4.1. Make the first part of assumptions of Theorem 2.6. Furthermore, assume that the joint performance process for the components of each module is associated in I. Then

$$1 - \bar{L}_\phi^{(I)} \geq 1 - \bar{L}_\psi[\bar{L}_{\chi_1}^{(I)},\ldots,\bar{L}_{\chi_r}^{(I)}]$$

$$L_\psi[L_{\chi_1}^{(I)},\ldots,L_{\chi_r}^{(I)}] \geq L_\phi^{(I)} .$$

This conjecture would then in turn give the following conjecture, which in a way is a generalization of the inequalities 1 and 3 of both Theorem 3.3 and 3.4.

Conjecture 4.2. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^{r}$ consisting of totally independent modules in the time interval I and with organizing coherent structure function ψ. Furthermore, for $i=1,\ldots,k$ $(1\leq k\leq r)$ assume that (A_i,χ_i) has a modular decomposition $\{(B_{ij},\Omega_{ij})\}_{j=1}^{s_i}$ consisting of totally independent modules in the time interval I and with organizing coherent structure function σ_i. Finally, assume that for each of the modules (B_{ij},Ω_{ij}) $i=1,\ldots,k\,;\,j=1,\ldots,s_i$ and (A_i,χ_i) $i=k+1,\ldots,r$ the joint performance process for the components is associated in I. Then

$$1-\bar{L}_{\psi}[g_{\chi_1}^{(I)},\ldots,g_{\chi_k}^{(I)},\bar{L}_{\chi_{k+1}}^{(I)},\ldots,\bar{L}_{\chi_r}^{(I)}]$$

$$\leq 1-\bar{L}_{\Theta}[g_{\Omega_{11}}^{(I)},\ldots,g_{\Omega_{ks_k}}^{(I)},\bar{L}_{\chi_{k+1}}^{(I)},\ldots,\bar{L}_{\chi_r}^{(I)}]$$

$$L_{\psi}[h_{\chi_1}^{(I)},\ldots,h_{\chi_k}^{(I)},L_{\chi_{k+1}}^{(I)},\ldots,L_{\chi_r}^{(I)}]$$

$$\geq L_{\Theta}[h_{\Omega_{11}}^{(I)},\ldots,h_{\Omega_{ks_k}}^{(I)},L_{\chi_{k+1}}^{(I)},\ldots,L_{\chi_r}^{(I)}],$$

where the coherent structure Θ is defined in Theorem 3.3.

The "proof" of Conjecture 4.2 is very parallel to the proof of Theorem 3.4 applying Theorem 2.7, Conjecture 4.1 and Corollary 2.5.

For the special case treated in Section 3, Conjecture 4.1 is proved in Appendix III. In Appendix IV we point out where this argument breaks down in the general case. It might, however, be possible to construct a proof along other lines.

Appendix I

PROOF OF THE INEQUALITIES 5,6,11,12 OF THEOREM 2.6

We start by proving the inequalities 6 and 11 for the special case that ψ is a parallel structure.

Lemma AI.1. Make the first part of assumptions of Theorem 2.6, but assume furthermore that ψ is a parallel structure. Then

$$u_{\phi}^{\prime\prime(I)} \leq 1 - \prod_{i=1}^{r} \bar{l}_{\chi_i}^{\prime\prime(I)}$$

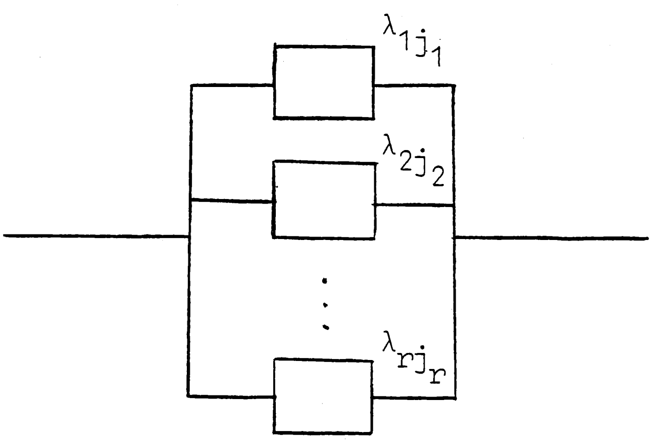
$$\bar{l}_{\phi}^{\prime\prime(I)} = \prod_{i=1}^{r} \bar{l}_{\chi_i}^{\prime\prime(I)} .$$

Proof: Let $(j_i=1,\ldots,m_i;i=1,\ldots,r)$

λ_{ij_i} = j_i-th minimal cut parallel structure of the i-th module

L_{ij_i} = j_i-th minimal cut set of the i-th module.

Now all minimal cut parallel structures for ϕ will be on the form



$1 \leq j_i \leq m_i$; $i=1,\ldots,r$

Hence

$$u''_\phi(I) = \min_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_r \le m_r}} P[\max_{\substack{k \in L_{ij_i} \\ i=1,\dots,r}} X_k(s) = 1 \quad \forall\, s \in \tau(I)]$$

$$= 1 - \max_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_r \le m_r}} P[\exists\, s \in \tau(I) \ni \max_{\substack{k \in L_{ij_i} \\ i=1,\dots,r}} X_k(s) = 0]$$

$$\le 1 - \max_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_r \le m_r}} P[\bigcap_{i=1}^{r} (X_k(s) = 0 \quad \forall\, k \in L_{ij_i}, \forall\, s \in \tau(I))]$$

$$= 1 - \max_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_r \le m_r}} \prod_{i=1}^{r} P[X_k(s) = 0 \quad \forall\, k \in L_{ij_i}, \forall\, s \in \tau(I)]$$

$$= 1 - \prod_{i=1}^{r} \bar{I}''_{X_i}(I) ,$$

having used the fact that the modules are totally independent in I. Finally

$$\bar{I}''_\phi(I) = \max_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_r \le m_r}} P[X_k(s) = 0 \quad \forall\, k \in L_{ij_i}; i=1,\dots,r, \forall\, s \in \tau(I)]$$

$$= \prod_{i=1}^{r} \bar{I}''_{X_i}(I),$$

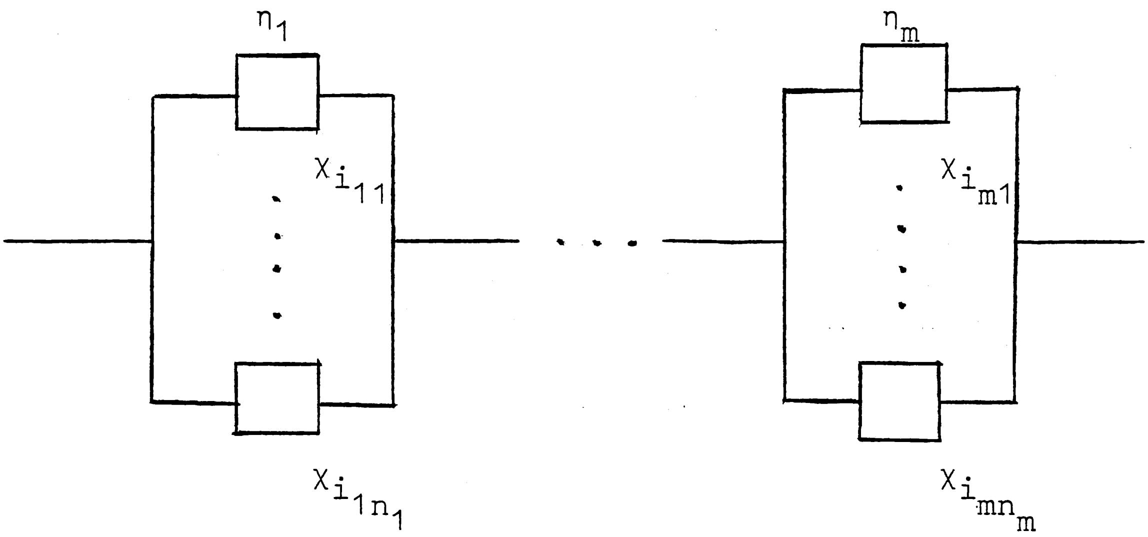
and the proof is completed.

Let now $(j=1,\ldots,m)$

η_j = j-th minimal cut parallel structure of ψ

E_j = j-th minimal cut set of ψ.

The structure (C,ϕ) can now graphically be represented by the modules as



where for $k \neq l$

$$\{i_{k1},\ldots,i_{kn_k}\} \neq \{i_{l1},\ldots,i_{ln_l}\}.$$

Define

$$\phi_j(\underline{x}) = \eta_j(\chi_1(\underline{x}),\ldots,\chi_r(\underline{x}))$$

and let $(k=1,\ldots,m_j;j=1,\ldots,m)$

μ_{jk} = k-th minimal cut parallel structure of $\phi_j(\underline{x})$

M_{jk} = k-th minimal cut set of $\phi_j(\underline{x})$.

As in the proof of Theorem 4.1 in Barlow and Proschan (1975) we realize that

$$\left\{\mu_{jk}\right\}_{j=1,\dots,m}^{k=1,\dots,m_j}$$

is the set of minimal cut parallel structures of ϕ. Hence

$$u''^{(I)}_{\phi} = \min_{1\leq j\leq m} \min_{1\leq k\leq m_j} P\left[\max_{l\in M_{jk}} X_l(s) = 1 \quad \forall s \in \tau(I)\right]$$

$$= \min_{1\leq j\leq m} u''^{(I)}_{\phi_j} \leq \min_{1\leq j\leq m} \left(1 - \prod_{i\in E_j} \bar{l}''^{(I)}_{\chi_i}\right)$$

$$= 1 - \max_{1\leq j\leq m} \prod_{i\in E_j} \bar{l}''^{(I)}_{\chi_i} = 1 - \bar{l}'_{\psi}\left(\bar{l}''^{(I)}_{\chi_1},\dots,\bar{l}''^{(I)}_{\chi_r}\right), \qquad \text{(AI.1)}$$

having applied Lemma AI.1 and the inequality 6 is proved. Furthermore

$$\bar{l}''^{(I)}_{\phi} = \max_{1\leq j\leq m} \max_{1\leq k\leq m_j} P\left[\max_{l\in M_{jk}} X_l(s) = 0 \quad \forall s \in \tau(I)\right]$$

$$= \max_{1\leq j\leq m} \bar{l}''^{(I)}_{\phi_j} = \max_{1\leq j\leq m} \prod_{i\in E_j} \bar{l}''^{(I)}_{\chi_i}$$

$$= \bar{l}'_{\psi}\left(\bar{l}''^{(I)}_{\chi_1},\dots,\bar{l}''^{(I)}_{\chi_r}\right), \qquad \text{(AI.2)}$$

again having applied Lemma AI.1 and the inequality 11 is proved.

Applying the inequality (AI.1) on the dual structure ϕ^D, we get

$$u''^{(I)}_{\phi^D} \leq 1 - \bar{l}'_{\psi^D}\left(\bar{l}''^{(I)}_{\chi_1^D},\dots,\bar{l}''^{(I)}_{\chi_r^D}\right),$$

which is quivalent to

$$\bar{u}''^{(I)}_{\phi} \leq 1 - l'_{\psi}\left(l''^{(I)}_{\chi_1},\dots,l''^{(I)}_{\chi_r}\right)$$

and the inequality 12 is proved. Similarly by applying the relation (AI.2) on the dual structure ϕ^D, we easily prove the inequality 5.

Appendix II

PROOF OF THE INEQUALITIES 1,6 OF THEOREM 2.7

We start by proving the inequality 1 for the special case that ψ is parallel structure.

Lemma AII.1. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^{r}$ and with organizing coherent structure function ψ being a parallel structure. Then

$$\bar{I}'_{\phi}(\underline{q}^{(I)}) = \prod_{i=1}^{r} \bar{I}'_{\chi_i}(\underline{q}^{(I)}) .$$

Proof: The proof is straightforward when remembering the notation of the proof of Lemma AI.1

$$\bar{I}'_{\phi}(\underline{q}^{(I)}) = \max_{\substack{1\le j_1\le m_1 \\ \vdots \\ 1\le j_r\le m_r}} \prod_{i=r}^{r} \prod_{k\in L_{ij_i}} q_k^{(I)}$$

$$= \prod_{i=1}^{r} \max_{1\le j_i\le m_i} \prod_{k\in L_{ij_i}} q_k^{(I)} = \prod_{i=1}^{r} \bar{I}'_{\chi_i}(\underline{q}^{(I)}) .$$

Now remembering the main part of the proof from Appendix I, we get by applying Lemma AII.1

$$\bar{I}'_{\phi}(\underline{q}^{(I)}) = \max_{1\le j\le m} \max_{1\le k\le m_j} \prod_{k\in M_{jk}} q_k^{(I)}$$

$$= \max_{1\le j\le m} \bar{I}'_{\phi_j}(\underline{q}^{(I)}) = \max_{1\le j\le m} \prod_{i\in E_j} \bar{I}'_{\chi_i}(\underline{q}^{(I)})$$

$$= \bar{I}'_{\psi}\left[\bar{I}'_{\chi_1}(\underline{q}^{(I)}),\ldots,\bar{I}'_{\chi_r}(\underline{q}^{(I)})\right], \qquad \text{(AII.1)}$$

and the inequality 1 is proved.

Applying the relation (AII.1) on the dual structure ϕ^D, we get

$$\bar{I}'_{\phi^D}(\underline{p}^{(I)}) = \bar{I}'_{\psi^D}\left[\bar{I}'_{\chi_1^D}(\underline{p}^{(I)}),\ldots,\bar{I}'_{\chi_r^D}(\underline{p}^{(I)}\right] ,$$

which is equivalent to

$$l'_{\phi}(\underline{p}^{(I)}) = l'_{\psi}\left[l'_{\chi_1}(\underline{p}^{(I)}),\ldots,l'_{\chi_r}(\underline{p}^{(I)})\right], \qquad \text{(AII.2)}$$

and the inequality 6 is proved.

Appendix III

PROOF OF THE INEQUALITIES 1, 10 OF THEOREM 3.2.

In order to establish the inequality 10 we first have to prove two lemmas.

Lemma AIII.1. Let (C,ϕ) be a coherent system with modular decomposition $\{(A_i,\chi_i)\}_{i=1}^{r}$ and with organizing coherent structure function ψ. Then

$$l'_{\phi}(\underline{p}) = l'_{\psi}\left[l'_{\chi_1}(\underline{p}^{A_1}),\ldots,l'_{\chi_r}(\underline{p}^{A_r})\right] .$$

Proof: The relation follows immediately from (AII.2) by specializing $I = [t,t]$.

Lemma AIII.2. Make the same assumptions as in Theorem 3.1. Then

$$l^*_{\phi} \leq l^*_{\psi}[l^*_{\chi_1},\ldots,l^*_{\chi_r}] .$$

Proof: We first prove the lemma for the case that ψ is a parallel structure; i.e. we will prove that

$$l^*_{\phi} \leq \coprod_{i=1}^{r} l^*_{\chi_i} \qquad \text{(AIII.1)}$$

Introduce the same notation as in the proof of Lemma AI.1. Consider the following structure

$$\phi^*(\underline{Y}(t)) = \prod_{i=1}^{r} \prod_{j_i=1}^{m_i} Y_{ij_i}(t),$$

where $\underline{Y}(t)$ is a vector of mutually independent (and hence associated) components at time t with reliabilities at this point of time $(j_i=1,\ldots,m_i;i=1,\ldots,r)$

$$P(Y_{ij_i}(t) = 1) = h_{\lambda_{ij_i}} .$$

We have

$$h_{\phi^*} = E\Big[\prod_{i=1}^{r} \prod_{j_i=1}^{m_i} Y_{ij_i}\Big] = \prod_{i=1}^{r} \prod_{j_i=1}^{m_i} h_{\lambda_{ij_i}} =$$

$$= \prod_{i=1}^{r} l^*_{\chi_i} \tag{AIII.2}$$

Now all minimal cut parallel structures, β_j, $j=1,\ldots,M$ for ϕ^* will be on the same form as for ϕ just replacing λ_{ij_i} by Y_{ij_i} $(j_i=1,\ldots,m_i;i=1,\ldots,r)$. (See the proof of Lemma AI.1.) Here

$$M = \prod_{i=1}^{r} m_i .$$

Since the modules of ϕ are totally independent at time t, $\lambda_{1j_1},\ldots,\lambda_{rj_r}$ are independent at this point of time, exactly as is true for $Y_{1j_1},\ldots,Y_{rj_r}$. We then have

$$l^*_{\phi^*} = \prod_{j=1}^{M} h_{\beta_j} = l^*_{\phi}.$$

Hence according to Theorem 3.4 of Barlow and Proschan (1975), which was generalized in Theorem 2.3, and (AIII.2) we have

$$l^*_\phi = l^*_{\phi*} \leq h_{\phi*} = \coprod_{i=1}^{r} l^*_{\chi_i},$$

and the first part of the proof is completed.

Now introduce the same notation as in the main part of the proof from Appendix I. We then have

$$l^*_\phi = \prod_{j=1}^{m} \prod_{k=1}^{m_i} h_{\mu_{jk}} = \prod_{j=1}^{m} l^*_{\phi_j}$$

$$\leq \prod_{j=1}^{m} \coprod_{i\in E_j} l^*_{\chi_i},$$

having applied (AIII.1), which is valid when the organizing structure function is a parallel structure. Furthermore,

$$\prod_{j=1}^{m} \coprod_{i\in E_j} l^*_{\chi_i} =$$

$$= l^*_\psi[l^*_{\chi_1},\ldots,l^*_{\chi_r}],$$

and the lemma is proved.

By applying Lemma AIII.1, we have

$$L_\psi[L_{\chi_1},\ldots,L_{\chi_r}] \geq l'_\psi\,[l'_{\chi_1}(\underline{p}^{A_1}),\ldots,l'_{\chi_r}(\underline{p}^{A_r})]$$

$$= l'_\phi(\underline{p}).$$

Furthermore from Lemma AIII.2

$$L_\psi[L_{\chi_1},\ldots,L_{\chi_r}] \geq l^*_\psi[l^*_{\chi_1},\ldots,l^*_{\chi_r}]$$

$$\geq l^*_\phi.$$

Hence the inequality 10 is proved.

Applying this inequality on the dual structure ϕ^D we get

$$L_{\phi^D} \leq L_{\psi^D}[L_{\chi_1^D},\ldots,L_{\chi_r^D}] ,$$

which is equivalent to

$$\bar{L}_{\phi} \leq \bar{L}_{\psi}[\bar{L}_{\chi_1},\ldots,\bar{L}_{\chi_r}],$$

and the inequality 1 is proved.

Appendix IV

SOME COMMENTS ON THE PROOF OF CONJECTURE 4.1

The natural way to prove Conjecture 4.1 is to generalize the argument given in the previous appendix. Having the relation (AII.1), Lemma AIII.1 is already generalized. It is Lemma AIII.2 that gives us the problems.

Introduce the structure $\phi^*(\underline{Y}(t))$ as in the latter lemma, where now the marginal performance processes $\{Y_{ij_i}(t),t\in\tau\}$ $j_i=1,\ldots,m_i$; $i=1,\ldots,r$ are independent in the time interval I, and where the probabilistic mechanism governing the marginal performance process $\{Y_{ij_i}(t),t\in\tau(I)\}$ is identical to the one governing $\{\lambda_{ij_i}(t),t\in\tau(I)\}$, $j_i=1,\ldots,m_i;i=1,\ldots,r$. Hence

$$l_{\phi}^{*(I)} = l_{\phi^*}^{*(I)} \leq h_{\phi^*}^{(I)} ,$$

having applied Corollary 2.3. This can be done since the marginal performance process for each λ_{ij_i} is associated in I and hence the same is true for Y_{ij_i}. It then follows that the joint performance process for the Y_{ij_i}-s is associated in I.

What remains is to establish

$$h_{\phi*}^{(I)} \leq \prod_{i=1}^{r} l_{\chi_i}^{*(I)} . \qquad \text{(AIV.1)}$$

We now have, introducing the sets S_m $m=1,2,\ldots$ as in the proof of Corollary 2.2

$$E\left[\prod_{l=1}^{m} \prod_{i=1}^{r} \prod_{j_i=1}^{m_i} Y_{ij_i}(s_l)\right]$$

$$= P\left[\forall s_l \ \exists\, i \ni \prod_{j_i=1}^{m_i} Y_{ij_i}(s_l) = 1\right]$$

$$\geq P\left[\exists i \ni \prod_{j_i=1}^{m_i} Y_{ij_i}(s_l) = 1 \ \forall\, s_l\right]$$

$$= E\left[\prod_{i=1}^{r} \prod_{j_i=1}^{m_i} \prod_{l=1}^{m} Y_{ij_i}(s_l)\right] = \prod_{i=1}^{r} \prod_{j_i=1}^{m_i} E\left[\prod_{l=1}^{m} Y_{ij_i}(s_l)\right] .$$

By letting $m \to \infty$, we get

$$h_{\phi*}^{(I)} \geq \prod_{i=1}^{r} \prod_{j_i=1}^{m_i} h_{\lambda_{ij_i}}^{(I)} = \prod_{i=1}^{r} l_{\chi_i}^{*(I)} .$$

This relation holds with equality iff $I = [t,t]$ without making further assumptions on the λ_{ij_i}-s. Hence we have not been able to establish (AIV.1), and our attempt to generalize Lemma AIII.2 did not succeed.

It should finally be noted that this is not the only way of proving Conjecture 4.1. If we for instance under certain assumptions could establish

$$l'_{\psi}\,[l_{\chi_1}^{*(I)},\ldots,l_{\chi_r}^{*(I)}] \geq l_{\phi}^{*(I)} ,$$

the conjecture follows immediately.

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